

STATE SPACE FORMULAS FOR A SUBOPTIMAL RATIONAL LEECH PROBLEM I: MAXIMUM ENTROPY SOLUTION

A.E. FRAZHO, S. TER HORST, AND M.A. KAASHOEK

ABSTRACT. For the strictly positive case (the suboptimal case) the maximum entropy solution X to the Leech problem $G(z)X(z) = K(z)$ and $\|X\|_\infty = \sup_{|z| \leq 1} \|X(z)\| \leq 1$, with G and K stable rational matrix functions, is proved to be a stable rational matrix function. An explicit state space realization for X is given, and $\|X\|_\infty$ turns out to be strictly less than one. The matrices involved in this realization are computed from the matrices appearing in a state space realization of the data functions G and K . A formula for the entropy of X is also given.

1. INTRODUCTION

Let G and K be matrix-valued H^∞ functions on the open unit disc \mathbb{D} of sizes $m \times p$ and $m \times q$, respectively, and let T_G and T_K denote the corresponding block lower triangular Toeplitz operators,

$$T_G : \ell_+^2(\mathbb{C}^p) \rightarrow \ell_+^2(\mathbb{C}^m), \quad T_K : \ell_+^2(\mathbb{C}^q) \rightarrow \ell_+^2(\mathbb{C}^m).$$

A $p \times q$ matrix-valued H^∞ function X is called a *solution to the Leech problem associated with G and K* whenever

$$(1.1) \quad G(z)X(z) = K(z) \quad (z \in \mathbb{D}) \quad \text{and} \quad \|X\|_\infty = \sup_{z \in \mathbb{D}} \|X(z)\| \leq 1.$$

The Leech problem is an example of a metric constrained interpolation problem, the first part of (1.1) is the interpolation condition, and the second part is the metric constraint. In a note dating from 1971/1972, only published recently [18], see also [17], Leech proved that the problem is solvable if and only if the operator $T_G T_G^* - T_K T_K^*$ is nonnegative. Later the Leech theorem was derived as a corollary of more general results; see, e.g., [19, page 107], [8, Section VIII.6]), and [2, Section 4.7].

Now assume in addition that G and K are rational. In other words, assume that G and K are stable rational matrix functions. In that case, if the Leech problem associated with G and K is solvable, one expects the problem to have a stable rational matrix solution as well. However, a priori this is not clear, and the existence of rational solutions was proved only recently in [20] by reducing the problem to polynomials, in [16] by adapting the lurking isometry method used in [3], and in [11] by using a state space approach.

In the present paper G and K are also stable rational matrix functions. We assume additionally that the operator $T_G T_G^* - T_K T_K^*$ is strictly positive. It is then known from commutant lifting theory that the Leech problem has a unique

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maximum entropy solution, that is, the (unique) solution X to the Leech problem associated with G and K for which the quantity

$$(1.2) \quad \mathcal{E}(X) = \frac{1}{2\pi} \int_0^{2\pi} \ln \det[I_q - X(e^{i\omega})^* X(e^{i\omega})] d\omega$$

is maximal. In this paper we show that this maximum entropy solution is a stable rational matrix function, we derive an explicit formula for this solution and a formula for its entropy $\mathcal{E}(X)$; see Theorem 1.2 below. When $T_G T_G^* - T_K T_K^*$ is only non-negative, the maximum entropy solution still exists but the problem whether or not it is rational remains open.

To prove the above mentioned results, we use the fact, well-known from mathematical systems theory (see, e.g., Chapter 1 of [7] or Chapter 4 in [4]), that rational matrix functions admit state space realizations. For our G and K this means that the matrix function $\begin{bmatrix} G & K \end{bmatrix}$ admits a representation of the following form:

$$(1.3) \quad \begin{bmatrix} G(z) & K(z) \end{bmatrix} = \begin{bmatrix} D_1 & D_2 \end{bmatrix} + zC(I_n - zA)^{-1} \begin{bmatrix} B_1 & B_2 \end{bmatrix}.$$

Here I_n is the $n \times n$ identity matrix, A is an $n \times n$ matrix, and B_1, B_2, C, D_1 and D_2 are matrices of appropriate sizes. Moreover, since G and K are stable rational matrix functions, G and K have no pole in the closed unit disc, and therefore we may assume that matrix A is *stable*, that is, A has all its eigenvalues in the open unit disc. The realization (1.3) is called *minimal* if there exists no realization of $\begin{bmatrix} G & K \end{bmatrix}$ as in (1.3) with ‘state matrix’ A of smaller size than the one in the given realization. In that case the order n of A is called the *McMillan degree* of $\begin{bmatrix} G & K \end{bmatrix}$. If the realization (1.3) is minimal, then the matrix A is automatically stable and the observability operator W_{obs} , which is defined by

$$(1.4) \quad W_{obs} = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \end{bmatrix} : \mathbb{C}^n \rightarrow \ell_+^2(\mathbb{C}^m),$$

is one-to-one. In the sequel we do not require the realization (1.3) to be minimal but we shall always assume that A is stable and W_{obs} is one-to-one. In that case we refer to (1.3) as an *observable stable realization*.

As a first step towards our main result we first derive, in Theorem 1.1 below, a necessary and sufficient condition for $T_G T_G^* - T_K T_K^*$ to be strictly positive in terms of the matrices in (1.3) and related matrices. To do this we need the rational $m \times m$ matrix function

$$(1.5) \quad R(z) = G(z)G^*(z) - K(z)K^*(z).$$

Here $G^*(z) = G(\bar{z}^{-1})^*$ and $K^*(z) = K(\bar{z}^{-1})^*$. Note that R has no pole on the unit circle \mathbb{T} . By T_R we denote the Toeplitz operator defined by R . Using the realization (1.3) one shows (see [11, Lemma 3.1]) that R admits the following state space representation:

$$(1.6) \quad R(z) = zC(I - zA)^{-1}\Gamma + R_0 + \Gamma^*(zI - A^*)^{-1}C^*.$$

Here R_0 and Γ are matrices of sizes $m \times m$ and $n \times m$, respectively, defined by

$$(1.7) \quad R_0 = D_1 D_1^* - D_2 D_2^* + C(P_1 - P_2)C^*,$$

$$(1.8) \quad \Gamma = B_1 D_1^* - B_2 D_2^* + A(P_1 - P_2)C^*,$$

and P_1 and P_2 are the unique $n \times n$ matrix solutions of the symmetric Stein equations:

$$(1.9) \quad P_1 - AP_1A^* = B_1B_1^* \quad \text{and} \quad P_2 - AP_2A^* = B_2B_2^*.$$

Since A is stable, the above equations are solvable and the solutions are unique. Finally, we associate with R the algebraic Riccati equation:

$$(1.10) \quad Q = A^*QA + (C - \Gamma^*QA)^*(R_0 - \Gamma^*Q\Gamma)^{-1}(C - \Gamma^*QA).$$

We are now ready to state our main results.

Theorem 1.1. *Let G and K be stable rational matrix functions, and assume that $\begin{bmatrix} G & K \end{bmatrix}$ is given by the observable stable realization (1.3). Then the operator $T_G T_G^* - T_K T_K^*$ is strictly positive if and only if the following two conditions hold.*

- (i) *There exists a strictly positive $n \times n$ matrix Q such that*
 - (a) *$R_0 - \Gamma^*Q\Gamma$ is strictly positive,*
 - (b) *Q satisfies the Riccati equation (1.10),*
 - (c) *the matrix $A_0 = A - \Gamma(R_0 - \Gamma^*Q\Gamma)^{-1}(C - \Gamma^*QA)$ is stable.*
- (ii) *The operator $Q^{-1} + P_2 - P_1$ is strictly positive.*

In this case, the Toeplitz operator T_R is strictly positive and the inverse of the operator $T_G T_G^ - T_K T_K^*$ is given by*

$$(1.11) \quad \left(T_G T_G^* - T_K T_K^* \right)^{-1} = T_R^{-1} + T_R^{-1} W_{obs} \Omega W_{obs}^* T_R^{-1}, \quad \text{where}$$

$$\Omega = (P_1 - P_2)(Q^{-1} + P_2 - P_1)^{-1} Q^{-1} = (P_1 - P_2)(I + Q(P_2 - P_1))^{-1}.$$

The second main result shows that the maximum entropy solution is rational and provides a state space realization for this solution.

Theorem 1.2. *Let G and K be stable rational matrix functions, and assume that $\begin{bmatrix} G & K \end{bmatrix}$ is given by the observable stable realization (1.3). Furthermore, assume that $T_G T_G^* - T_K T_K^*$ is strictly positive, or equivalently, that items (i) and (ii) of Theorem 1.1 hold. Then the maximal entropy solution X to the Leech problem for G and K is a stable rational matrix function which is given by the following state space realization:*

$$(1.12) \quad X(z) = D_U D_V^{-1} + z (C_1 - D_U D_V^{-1} C_2) (I - z A^\times)^{-1} B_0 D_V^{-1}.$$

Here, using the matrices appearing in Theorem 1.1, the matrices in (1.12) are defined by

$$(1.13) \quad \Delta = R_0 - \Gamma^*Q\Gamma, \quad C_0 = \Delta^{-1}(C - \Gamma^*QA), \quad A_0 = A - \Gamma C_0;$$

$$C_j = D_j^* C_0 + B_j^* Q A_0, \quad (j = 1, 2);$$

$$(1.14) \quad D_0 = \Delta^{-1}(D_2 - \Gamma^*QB_2) + C_0 \Omega C_2^*;$$

$$(1.15) \quad B_0 = B_2 - \Gamma \Delta^{-1}(D_2 - \Gamma^*QB_2) + A_0 \Omega C_2^*;$$

$$(1.16) \quad D_U = D_1^* D_0 + B_1^* Q B_0, \quad D_V = I_q + D_2^* D_0 + B_2^* Q B_0;$$

$$A^\times = A_0 - B_0 D_V^{-1} C_2.$$

Moreover, the state matrix A^\times is stable, the matrix D_V is strictly positive, and the entropy of X is given by

$$(1.17) \quad \mathcal{E}(X) = -\ln \det[D_V].$$

Finally, $\|X\|_\infty = \sup_{|z| \leq 1} \|X(z)\|$ is strictly less than one, and the McMillan degree of X is less than or equal to the McMillan degree of $[G \ K]$.

A description of all solutions to the Leech problem (1.1) for the case when G and K are rational will be the topic of a future publication.

Remark 1.3. An $n \times n$ matrix Q is said to be a *stabilizing solution* of the algebraic Riccati equation (1.10) whenever Q satisfies the three conditions (a), (b) and (c) in item (i) of Theorem 1.1. In this case Q is uniquely determined, cf., formula (1.18) below. Moreover, the existence of a stabilizing solution of (1.10) is equivalent to the Toeplitz operator T_R being strictly positive. In that case, the stabilizing matrix Q is given by

$$(1.18) \quad Q = W_{obs}^* T_R^{-1} W_{obs}.$$

See, e.g., [10, Section 10.3], [5, Section 14.7], and [12, Theorem 1.1] for a non-symmetric version. Also note that there exist several efficient numerical algorithms to compute a stabilizing solution, cf., [1].

The special case of Leech's theorem with $q = m$ and K identically equal to the $m \times m$ identity matrix I_m is part of the corona theorem, which is due to Carlson [6], for $m = 1$, and Fuhrmann [15] for arbitrary m . The least squares solution of the corona version of the equation can be found in [13] and a description of all solutions without any norm constraint in [14]. For an engineering perspective on corona and Leech type problems and related applications in signal processing we refer to [22, 21] and the references therein.

The paper consists of five sections including the present introduction. In Section 2 we recall the basic results from commutant lifting theory used in the present paper, and we specify these results for the Leech problem. In Section 3 we assume that $T_G T_G^* - T_K T_K^*$ is strictly positive and, using the commutant lifting results, we derive an infinite dimensional state space realization for the maximum entropy solution. These two sections do not require G and K to be rational; the next two sections do. In Section 4 we further clarify the role of the Toeplitz operator T_R with R being given by (1.5) and prove Theorem 1.1. The proof of Theorem 1.2 is given in the final section. At the end of the final section we present a direct proof of the fact that the function X given by (1.12) satisfies the first identity in (1.1) (see Remark 5.3).

Some terminology and notation. For any positive integer k we write E_k for the canonical embedding of \mathbb{C}^k onto the first coordinate space of $\ell_+^2(\mathbb{C}^k)$, that is,

$$E_k = [I_k \ 0 \ 0 \ 0 \ \dots]^\top : \mathbb{C}^k \rightarrow \ell_+^2(\mathbb{C}^k).$$

Here $\ell_+^2(\mathbb{C}^k)$ denotes the Hilbert space of unilateral square summable sequences of vectors in \mathbb{C}^k . By S_k we denote the unilateral shift on $\ell_+^2(\mathbb{C}^k)$. For positive integers k and r we write $H_{k \times r}^\infty$ for the Banach space of all $k \times r$ matrices with entries from H^∞ , the algebra of all bounded analytic functions of the open unit disc \mathbb{D} . As usual, we identify a $k \times r$ matrix with complex entries with the linear operator from \mathbb{C}^r to \mathbb{C}^k induced by the action of the matrix on the standard bases. By definition, the infinity norm of $F \in H_{k \times r}^\infty$ is given by $\|F\|_\infty = \sup_{|z| < 1} \|F(z)\|$. A function $F \in H_{k \times r}^\infty$ is said to be *outer* if the Toeplitz operator T_F from $\ell_+^2(\mathbb{C}^r)$ to $\ell_+^2(\mathbb{C}^k)$ defined by F has a dense range. We call $F \in H_{k \times k}^\infty$ *invertible outer* if $\det F(z) \neq 0$

for each $z \in \mathbb{D}$ and F^{-1} belongs to $H_{k \times k}^\infty$. Thus $F \in H_{k \times k}^\infty$ is invertible outer if and only if T_F is invertible, and in that case $T_F^{-1} = T_{F^{-1}}$.

2. THE CENTRAL COMMUTANT LIFTING SOLUTION

In this section we recall the construction of the central solution in the Sz.-Nagy-Foias commutant lifting theorem, as presented in Chapter IV of [9] with the bound γ equal to one. Note that in this setting, by [9, Theorem IV.7.5], the central solution is equal to the maximum entropy solution (see Theorem 2.2 below).

Theorem 2.1 (Commutant lifting). *Let \mathcal{H}' be an invariant subspace for the backward shift S_p^* on $\ell_+^2(\mathbb{C}^p)$ and T' the operator on \mathcal{H}' obtained by compressing S_p to \mathcal{H}' , that is, $T' = P_{\mathcal{H}'} S_p|_{\mathcal{H}'}$. Let Λ be a contraction mapping $\ell_+^2(\mathbb{C}^q)$ into \mathcal{H}' satisfying $T' \Lambda = \Lambda S_q$. Then there exists a function X in $H_{p \times q}^\infty$ such that*

$$(2.1) \quad \Lambda = P_{\mathcal{H}'} T_X \quad \text{and} \quad \|X\|_\infty \leq 1.$$

Moreover, if $\|\Lambda\| < 1$, then a function X in $H_{p \times q}^\infty$ satisfying $\Lambda = P_{\mathcal{H}'} T_X$ and $\|X\|_\infty \leq 1$ is given by

$$(2.2) \quad \begin{aligned} X(z) &= U(z) V(z)^{-1}, \\ U(z) &= E_p^* (I - z S_p^*)^{-1} \Lambda (I - \Lambda^* \Lambda)^{-1} E_q, \\ V(z) &= E_q^* (I - z S_q^*)^{-1} (I - \Lambda^* \Lambda)^{-1} E_q. \end{aligned}$$

Moreover, $\det V(z) \neq 0$ for $|z| < 1$, the function V^{-1} belongs to $H_{q \times q}^\infty$ and is an outer function. In fact, the function $\Theta = V(0)^{1/2} V^{-1}$ is the outer spectral factor of the function $I - X^* X$, that is

$$(2.3) \quad I - X(\zeta)^* X(\zeta) = \Theta(\zeta)^* \Theta(\zeta), \quad \zeta \in \mathbb{T} \quad \text{a.e.}$$

The formulas for X , U and V appearing in (2.2) and the identity (2.3) are obtained from [9, Theorem IV.6.6] using $\gamma = 1$ and $A = \Lambda$.

The following theorem (see [9, Theorem IV.7.5]) shows that the function X constructed in the second part of the above theorem is the maximum entropy solution.

Theorem 2.2. *The function X in (2.2) is the maximal entropy solution, that is, if Y in $H_{p \times q}^\infty$ satisfies $\Lambda = P_{\mathcal{H}'} T_Y$ and $\|Y\|_\infty \leq 1$, then $\mathcal{E}(Y) \leq \mathcal{E}(X)$. Moreover, the maximal entropy solution is unique and*

$$(2.4) \quad \mathcal{E}(X) = -\ln \det[V(0)] = -\ln \det[E_q^* (I - \Lambda^* \Lambda)^{-1} E_q].$$

In the remaining part of this section we will apply the previous theorems to the special choice of Λ associated with our Leech problem. For this special case Λ is given in item (iii) of the following lemma for the general case when G and K are matrix-valued H^∞ functions and not necessarily rational functions.

Lemma 2.3. *Let G and K be matrix-valued H^∞ functions of sizes $m \times p$ and $m \times q$, respectively, and assume that $T_G T_G^* - T_K T_K^*$ is strictly positive. Then the following statements hold.*

- (i) *The operator $T_G T_G^*$ is invertible, or equivalently, T_G^* is one-to-one and has closed range.*
- (ii) *The subspace $\mathcal{H}' = \text{Im } T_G^*$ is invariant for the backward shift S_p^* .*

- (iii) The operator $\Lambda = T_G^* (T_G T_G^*)^{-1} T_K$ viewed as an operator from $\ell_+^2(\mathbb{C}^q)$ into \mathcal{H}' is a strict contraction. Moreover,

$$(2.5) \quad T_G \Lambda = T_K.$$

- (iv) The operator Λ intertwines S_q with T' , that is,

$$(2.6) \quad T' \Lambda = \Lambda S_q$$

where T' on \mathcal{H}' is the compression of S_p to \mathcal{H}' , that is, $T' = P_{\mathcal{H}'} S_p|_{\mathcal{H}'}$.

Proof. Because $T_G T_G^* - T_K T_K^*$ is strictly positive, $T_G T_G^*$ is also strictly positive. Hence T_G is one-to-one and has closed range. Thus item (i) holds. From item (i) we conclude that \mathcal{H}' is a closed subspace of $\ell_+^2(\mathbb{C}^p)$. Using $S_p^* T_G^* = T_G^* S_m^*$, it follows that $\mathcal{H}' = \text{Im } T_G^*$ is an invariant subspace for the backward shift S_p^* . Therefore item (ii) holds.

Using the definition of Λ we see that

$$T_G \Lambda u = T_G T_G^* (T_G T_G^*)^{-1} T_K u = T_K u, \quad u \in \ell_+^2(\mathbb{C}^q).$$

This proves (2.5). We also have $\Lambda^* T_G^* = T_K^*$. It follows that the operator

$$T_G(I - \Lambda \Lambda^*) T_G^* = T_G T_G^* - T_K T_K^*$$

is strictly positive. Using $\mathcal{H}' = \text{Im } T_G^*$ and the fact that T_G^* is one-to-one and has closed range, we conclude that $I - \Lambda \Lambda^*$ is also strictly positive. In other words, Λ is a strict contraction and item (iii) holds.

Recall that T' is the compression of S_p onto \mathcal{H}' . Because \mathcal{H}' is an invariant subspace for the backward shift S_p^* , we have $P_{\mathcal{H}'} S_p = T' P_{\mathcal{H}'}$. (In the language of the commutant lifting theorem, S_p is an isometric dilation of T' .) Notice that $\mathcal{H}' = (\text{Ker } T_G)^\perp$. Now observe that

$$T_G T' \Lambda = T_G S_p \Lambda = S_m T_G \Lambda = S_m T_K = T_K S_q = T_G \Lambda S_q.$$

Hence $T' \Lambda = \Lambda S_q$. Therefore item (iv) holds. \square

The above lemma shows that operator $\Lambda = T_G^* (T_G T_G^*)^{-1} T_K$ mapping $\ell_+^2(\mathbb{C}^q)$ into \mathcal{H}' satisfies the hypothesis of Theorem 2.1. In particular, $X = UV^{-1}$ in (2.2) is the maximal entropy solution; see Theorem 2.2. In Section 5, we will construct the finite dimensional state space realization for X in Theorem 1.2. According to Theorem 2.1 the operator $\Lambda = P_{\mathcal{H}'} T_X$ and $\|X\|_\infty \leq 1$. Hence $T_K = T_G \Lambda = T_G T_X$, or equivalently, $GX = K$. Therefore X is a solution to the Leech problem associated to G and K . This $X = UV^{-1}$ is also the unique maximal entropy solution over the set of all contractive analytic solutions Y for $GY = K$. If Y is contractive analytic solution to $GY = K$, then $T_G T_Y = T_K$ and $\|T_Y\| = \|Y\|_\infty \leq 1$. Notice that

$$T_G P_{\mathcal{H}'} T_Y = T_G T_Y = T_K = T_G \Lambda.$$

Because $\mathcal{H}' = (\text{Ker } T_G)^\perp$, we have $\Lambda = P_{\mathcal{H}'} T_Y$. Theorem 2.2 guarantees that $\mathcal{E}(Y) \leq \mathcal{E}(X)$ with equality if and only if $Y = X$. Since $X = UV^{-1}$, formula (1.17) is a direct consequence of Theorem 2.2.

3. THE INFINITE DIMENSIONAL STATE SPACE MODEL

Throughout this section $G \in H_{m \times p}^\infty$ and $K \in H_{m \times q}^\infty$, and $T_G T_G^* - T_K T_K^*$ is assumed to be strictly positive. Furthermore, $\Lambda = T_G^* (T_G T_G^*)^{-1} T_K$ is viewed as an operator from $\ell_+^2(\mathbb{C}^q)$ into $\mathcal{H}' = \text{Im } T_G^*$, and $X = UV^{-1}$ is the maximal entropy solution in Theorem 2.1 corresponding to this choice of Λ .

The following proposition provides an infinite dimensional state space realization for X even when G and K are nonrational.

Proposition 3.1. *Assume that $T_G T_G^* - T_K T_K^*$ is strictly positive, where G and K are functions in $H_{m \times p}^\infty$ and $H_{m \times q}^\infty$, respectively. Let Λ be the strict contraction defined by $\Lambda = T_G^* (T_G T_G^*)^{-1} T_K$. Then the function $X = UV^{-1}$ in (2.2), is given by the following infinite dimensional state space realization*

$$(3.1) \quad X(z) = D_U D_V^{-1} + z (E_p^* T_G^* - D_U D_V^{-1} E_q^* T_K^*) (I - zF)^{-1} S_m^* \Xi D_V^{-1} \quad (z \in \mathbb{D}).$$

Here Ξ and F are the operators defined by

$$(3.2) \quad \Xi = (T_G T_G^* - T_K T_K^*)^{-1} T_K E_q : \mathbb{C}^q \rightarrow \ell_+^2(\mathbb{C}^m),$$

$$(3.3) \quad F = S_m^* - S_m^* \Xi D_V^{-1} E_q^* T_K^* \text{ on } \ell_+^2(\mathbb{C}^m),$$

and D_U and D_V are given by

$$(3.4) \quad D_U = E_p^* T_G^* \Xi : \mathbb{C}^q \rightarrow \mathbb{C}^p, \quad D_V = I_q + E_q^* T_K^* \Xi : \mathbb{C}^q \rightarrow \mathbb{C}^q.$$

Finally, the spectral radius $r_{\text{spec}}(F) \leq 1$.

The following lemma is used to prove the above result.

Lemma 3.2. *Assume that $T_G T_G^* - T_K T_K^*$ is strictly positive, where G and K are functions in $H_{m \times p}^\infty$ and $H_{m \times q}^\infty$, respectively. Let Λ be the strict contraction defined by $\Lambda = T_G^* (T_G T_G^*)^{-1} T_K$. Then*

$$(3.5) \quad (I - \Lambda^* \Lambda)^{-1} = I + T_K^* (T_G T_G^* - T_K T_K^*)^{-1} T_K,$$

$$(3.6) \quad \Lambda(I - \Lambda^* \Lambda)^{-1} = T_G^* (T_G T_G^* - T_K T_K^*)^{-1} T_K.$$

Proof. Using $\Lambda = T_G^* (T_G T_G^*)^{-1} T_K$, we obtain

$$I - \Lambda^* \Lambda = I - T_K^* (T_G T_G^*)^{-1} T_K.$$

The operator inversion formula $(I - C^* A^{-1} C)^{-1} = I + C^* (A - C C^*)^{-1} C$ yields the formula for $(I - \Lambda^* \Lambda)^{-1}$ in (3.5).

On the other hand,

$$\begin{aligned} \Lambda(I - \Lambda^* \Lambda)^{-1} &= T_G^* (T_G T_G^*)^{-1} T_K \left(I + T_K^* (T_G T_G^* - T_K T_K^*)^{-1} T_K \right) \\ &= T_G^* (T_G T_G^*)^{-1} \left(I + T_K T_K^* (T_G T_G^* - T_K T_K^*)^{-1} \right) T_K \\ &= T_G^* (T_G T_G^*)^{-1} \left((T_G T_G^* - T_K T_K^*) + T_K T_K^* \right) (T_G T_G^* - T_K T_K^*)^{-1} T_K \\ &= T_G^* (T_G T_G^* - T_K T_K^*)^{-1} T_K. \end{aligned}$$

This yields the formula for $\Lambda(I - \Lambda^* \Lambda)^{-1}$ in (3.6). \square

Proof of Proposition 3.1. Recall that $X = UV^{-1}$. We first establish a state space realization for U . By employing (3.6), we obtain

$$\begin{aligned} U(z) &= E_p^*(I - zS_p^*)^{-1}\Lambda(I - \Lambda^*\Lambda)^{-1}E_q \\ &= E_p^*(I - zS_p^*)^{-1}T_G^*(T_G T_G^* - T_K T_K^*)^{-1}T_K E_q \\ &= E_p^*T_G^*(I - zS_m^*)^{-1}(T_G T_G^* - T_K T_K^*)^{-1}T_K E_q \\ &= E_p^*T_G^*(I - zS_m^*)^{-1}\Xi, \quad |z| < 1. \end{aligned}$$

See (3.2) for the definition of Ξ . Using $D_U = E_p^*T_G^*\Xi$, we see that a state space realization for U is given by

$$(3.7) \quad U(z) = D_U + zE_p^*T_G^*(I - zS_m^*)^{-1}S_m^*\Xi, \quad |z| < 1.$$

To compute a state space realization for V , we use (3.5) in the following calculation:

$$\begin{aligned} V(z) &= E_q^*(I - zS_q^*)^{-1}(I - \Lambda^*\Lambda)^{-1}E_q \\ &= E_q^*E_q + E_q^*(I - zS_q^*)^{-1}T_K^*(T_G T_G^* - T_K T_K^*)^{-1}T_K E_q \\ &= I_q + E_q^*T_K^*(I - zS_m^*)^{-1}\Xi \\ &= I_q + E_q^*T_K^*\Xi + zE_q^*T_K^*(I - zS_m^*)^{-1}S_m^*\Xi, \quad |z| < 1. \end{aligned}$$

By consulting (3.4), we see that a state space realization for V is given by

$$(3.8) \quad V(z) = D_V + zE_q^*T_K^*(I - zS_m^*)^{-1}S_m^*\Xi, \quad |z| < 1.$$

Using a classical state space inversion formula, the inverse of $V(z)^{-1}$ in a neighborhood of zero is given by

$$\begin{aligned} V(z)^{-1} &= D_V^{-1} - zD_V^{-1}E_q^*T_K^*(I - zF)^{-1}S_m^*\Xi D_V^{-1}, \quad \text{where} \\ F &= S_m^* - S_m^*\Xi D_V^{-1}E_q^*T_K^*, \quad \text{as in (3.3)}. \end{aligned}$$

On the other hand, by the final part of Theorem 2.1, we know that $V(z)$ is invertible for each z in \mathbb{D} . Since $r_{\text{spec}}(S_m^*) \leq 1$, we can then apply Theorem 2.1 in [4] (with $\lambda = z^{-1}$) to show that $r_{\text{spec}}(F) \leq 1$. Thus

$$(3.9) \quad V(z)^{-1} = D_V^{-1} - zD_V^{-1}E_q^*T_K^*(I - zF)^{-1}S_m^*\Xi D_V^{-1}, \quad |z| < 1.$$

To compute a state space realization for the maximum entropy solution X , we first observe (using the identity (3.3)) that

$$\begin{aligned} &-zE_p^*T_G^*(I - zS_m^*)^{-1}S_m^*\Xi D_V^{-1}E_q^*T_K^*(I - zF)^{-1}S_m^*\Xi D_V^{-1} \\ &= zE_p^*T_G^*(I - zS_m^*)^{-1}(F - S_m^*)(I - zF)^{-1}S_m^*\Xi D_V^{-1} \\ &= E_p^*T_G^*(I - zS_m^*)^{-1}((I - zS_m^*) - (I - zF))(I - zF)^{-1}S_m^*\Xi D_V^{-1} \\ &= E_p^*T_G^*(I - zF)^{-1}S_m^*\Xi D_V^{-1} - E_p^*T_G^*(I - zS_m^*)^{-1}S_m^*\Xi D_V^{-1}. \end{aligned}$$

This readily implies that

$$\begin{aligned} &-z^2E_p^*T_G^*(I - zS_m^*)^{-1}S_m^*\Xi D_V^{-1}E_q^*T_K^*(I - zF)^{-1}S_m^*\Xi D_V^{-1} = \\ &= zE_p^*T_G^*(I - zF)^{-1}S_m^*\Xi D_V^{-1} - zE_p^*T_G^*(I - zS_m^*)^{-1}S_m^*\Xi D_V^{-1}. \end{aligned}$$

The state space realizations for U in (3.7) and V^{-1} in (3.9) then yield:

$$\begin{aligned} U(z)V(z)^{-1} &= D_U D_V^{-1} - z D_U D_V^{-1} E_q^* T_K^* (I - zF)^{-1} S_m^* \Xi D_V^{-1} \\ &\quad + z E_p^* T_G^* (I - zS_m^*)^{-1} S_m^* \Xi D_V^{-1} \\ &\quad - z^2 E_p^* T_G^* (I - zS_m^*)^{-1} S_m^* \Xi D_V^{-1} E_q^* T_K^* (I - zF)^{-1} S_m^* \Xi D_V^{-1} \\ &= D_U D_V^{-1} + z (E_p^* T_G^* - D_U D_V^{-1} E_q^* T_K^*) (I - zF)^{-1} S_m^* \Xi D_V^{-1} \end{aligned}$$

when $z \in \mathbb{D}$. This proves the state space formula for X in (3.1). \square

4. PROOF OF THEOREM 1.1

Throughout the section G and K are stable rational matrix functions of sizes $m \times p$ and $m \times q$, respectively, and we assume that $\begin{bmatrix} G & K \end{bmatrix}$ is given by the observable stable realization (1.3). We first prove two lemmas. The first deals with the $m \times m$ rational matrix function R defined by (1.5).

Lemma 4.1. *Let R be the $m \times m$ rational matrix function defined by (1.5). Then T_R is strictly positive whenever $T_G T_G^* - T_K T_K^*$ is strictly positive.*

Proof. Assume that $T_G T_G^* - T_K T_K^*$ is strictly positive. For each $z \in \mathbb{D}$ put $\varphi_{z,m} = \begin{bmatrix} I_m & zI_m & z^2 I_m & \cdots \end{bmatrix}^*$. Note that

$$T_G^* \varphi_{z,m} = \varphi_{z,p} G(z)^*, \quad T_K^* \varphi_{z,m} = \varphi_{z,q} K(z)^*, \quad \varphi_{z,m}^* \varphi_{z,m} = \frac{1}{1 - |z|^2} I_m.$$

Since $T_G T_G^* - T_K T_K^*$ is assumed to be strictly positive, there exists an $\eta > 0$ such that $T_G T_G^* - T_K T_K^* \geq \eta I$. Multiplying this inequality by $\varphi_{z,m}$ on the right and by $\varphi_{z,m}^*$ on the left gives

$$\frac{G(z)G(z)^* - K(z)K(z)^*}{1 - |z|^2} \geq \frac{\eta}{1 - |z|^2} I_m \quad (z \in \mathbb{D}).$$

Multiplying with $1 - |z|^2$ and taking limits $z \rightarrow e^{i\omega}$ on the unit circle, shows

$$R(e^{i\omega}) = G(e^{i\omega})G(e^{i\omega})^* - K(e^{i\omega})K(e^{i\omega})^* \geq \eta I_m, \quad 0 \leq \omega \leq 2\pi.$$

This implies $T_R \geq \eta I_m$. \square

Lemma 4.2. *Let W_{obs} be defined by (1.4), and let P_1 and P_2 be the unique $n \times n$ matrix solutions of the Stein equations (1.9). Then*

$$(4.1) \quad T_G T_G^* - T_K T_K^* = T_R + W_{obs}(P_2 - P_1)W_{obs}^*.$$

In particular, the operator $T_G T_G^ - T_K T_K^*$ is strictly positive if and only if the operator $T_R + W_{obs}(P_2 - P_1)W_{obs}^*$ is strictly positive.*

Proof. We first recall some elementary facts concerning Hankel operators. To this end, let

$$H_G = \begin{bmatrix} G_1 & G_2 & G_3 & \cdots \\ G_2 & G_3 & G_4 & \cdots \\ G_4 & G_5 & G_6 & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix} : \ell_+^2(\mathbb{C}^p) \rightarrow \ell_+^2(\mathbb{C}^m)$$

be the Hankel operator determined by the Taylor series $G(z) = \sum_{\nu=0}^{\infty} z^\nu G_\nu$. In a similar way, let H_K be the corresponding Hankel operator mapping $\ell_+^2(\mathbb{C}^q)$ into

$\ell_+^2(\mathbb{C}^m)$ determined by K . Let $W_{con,1}$ mapping $\ell_+^2(\mathbb{C}^p)$ into \mathbb{C}^n and $W_{con,2}$ mapping $\ell_+^2(\mathbb{C}^q)$ into \mathbb{C}^n be the controllability operators defined by

$$W_{con,j} = \begin{bmatrix} B_j & AB_j & A^2B_j & A^3B_j & \cdots \end{bmatrix}, \quad j = 1, 2.$$

From (1.9) we see that $P_j = W_{con,j}W_{con,j}^*$ for $j = 1, 2$. Using $G_\nu = CA^{\nu-1}B_1$ for all integers $\nu \geq 1$ and the corresponding result for K , we see that $H_G = W_{obs}W_{con,1}$ and $H_K = W_{obs}W_{con,1}$. Finally,

$$(4.2) \quad H_G H_G^* = W_{obs} P_1 W_{obs}^* \quad \text{and} \quad H_K H_K^* = W_{obs} P_2 W_{obs}^*.$$

Next, notice the Toeplitz operators T_{GG^*} and T_{KK^*} are given by the following identities:

$$T_{GG^*} = T_G T_G^* + H_G H_G^* \quad \text{and} \quad T_{KK^*} = T_K T_K^* + H_K H_K^*.$$

Using $R = GG^* - KK^*$, we have $T_R = T_G T_G^* - T_K T_K^* + H_G H_G^* - H_K H_K^*$. But then (4.2) yields (4.1). \square

Proof of Theorem 1.1. Assume the operator $T_G T_G^* - T_K T_K^*$ is strictly positive. Then Lemma 4.1 tells us T_R is strictly positive, and hence, see Remark 1.3, item (i) in Theorem 1.1 is fulfilled. Furthermore, applying Lemma 4.3 below with

$$(4.3) \quad M = T_G T_G^* - T_K T_K^*, \quad T = T_R, \quad W = W_{obs} \quad \text{and} \quad N = P_2 - P_1,$$

noting that $M = T + WNW^*$ is strictly positive, by the identity (4.1), we see that the matrix $Q^{-1} + P_2 - P_1$ is strictly positive, and hence item (ii) in Theorem 1.1 is fulfilled. Furthermore, again in view of (4.1), in this case the inversion formula (4.5) yields the formula to compute the inverse of $T_G T_G^* - T_K T_K^*$ in (1.11).

Conversely, assume items (i) and (ii) in Theorem 1.1 are satisfied. Then item (i) implies that T_R is strictly positive, as explained in Remark 1.3, and $Q = W_{obs}^* T_R^{-1} W_{obs} = W^* T^{-1} W$, using the notation of (4.3) in the last identity. Note that item (ii) states that $Q^{-1} - N = Q^{-1} - P_1 + P_2$ is strictly positive. Hence again using Lemma 4.3 below and the identity (4.1), we see that item (ii) implies that $T_G T_G^* - T_K T_K^*$ is strictly positive. \square

Lemma 4.3. *Let M be an operator acting on a Hilbert space \mathcal{H} such that*

$$(4.4) \quad M = T + WNW^*,$$

where T on \mathcal{H} is a strictly positive operator, N is a self adjoint operator on a Hilbert space \mathcal{X} , and W is an operator mapping \mathcal{X} into \mathcal{H} which is one-to-one and has closed range. Set $Q = W^ T^{-1} W$. Then Q is invertible. Furthermore, M is strictly positive if and only if $Q^{-1} + N$ is strictly positive. Moreover, in that case,*

$$(4.5) \quad M^{-1} = T^{-1} - T^{-1} W N (I + Q N)^{-1} W^* T^{-1}.$$

Proof. Replacing M by $T^{-1/2} M T^{-1/2}$ and W by $T^{-1/2} W$, we see that without loss of generality we may assume that T is the identity operator on \mathcal{H} . Therefore, in what follows $M = I + WNW^*$. Note that in this case $Q = W^* W$.

The fact that W is one-to-one and has closed range, implies that $Q = W^* W$ is invertible. It follows that the Moore-Penrose left inverse W^+ of W is well-defined and is given by $W^+ = (W^* W)^{-1} W^* = Q^{-1} W^*$. Furthermore, the orthogonal project \mathbb{P} on \mathcal{H} mapping \mathcal{H} onto the range of W is given $\mathbb{P} = W Q^{-1} W^*$. Now note that

$$(4.6) \quad M = I + WNW^* = I - \mathbb{P} + \mathbb{P} + WNW^* = I - \mathbb{P} + W(Q^{-1} + N)W^*.$$

Put $\mathcal{H}_1 = \text{Im } W$ and $\mathcal{H}_0 = \text{Ker } W^*$, and consider the operators

$$\begin{aligned}\tau_1 : \mathcal{H}_1 &\rightarrow \mathcal{H}, \quad \tau_1 u = u \quad (u \in \mathcal{H}_1), \\ W_1 : \mathcal{X} &\rightarrow \mathcal{H}_1, \quad W_1 x = Wx \quad (x \in \mathcal{X}).\end{aligned}$$

Note that $W = \tau_1 W_1$ and $W^* = W_1^* \tau_1^*$. Furthermore, W_1 is invertible, and $W_1^{-1} = Q^{-1} W^* \tau_1$. Using (4.6) we see that relative to the orthogonal decomposition $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ the operator M admits the following block operator matrix representation:

$$M = \begin{bmatrix} I_{\mathcal{H}_0} & 0 \\ 0 & W_1(Q^{-1} + N)W_1^* \end{bmatrix}$$

Since W_1 is invertible, it follows that M is strictly positive if and only $Q^{-1} + N$ is strictly positive. Moreover, in that case

$$\begin{aligned}M^{-1} &= \begin{bmatrix} I_{\mathcal{H}_0} & 0 \\ 0 & W_1^{-*}(Q^{-1} + N)^{-1}W_1^{-1} \end{bmatrix} \\ &= \begin{bmatrix} I_{\mathcal{H}_0} & 0 \\ 0 & \tau_1^* W Q^{-1}(Q^{-1} + N)^{-1}Q^{-1}W^* \tau_1 \end{bmatrix}\end{aligned}$$

It follows that

$$\begin{aligned}M^{-1} &= I - \mathbb{P} + WQ^{-1}(Q^{-1} + N)^{-1}Q^{-1}W^* \\ &= I - WQ^{-1}W^* + WQ^{-1}(Q^{-1} + N)^{-1}Q^{-1}W^* \\ &= I - W\left(Q^{-1} - Q^{-1}(Q^{-1} + N)^{-1}Q^{-1}\right)W^*\end{aligned}$$

Finally, note that

$$\begin{aligned}Q^{-1} - Q^{-1}(Q^{-1} + N)^{-1}Q^{-1} &= \\ &= Q^{-1} - (Q^{-1} + N - N)(Q^{-1} + N)^{-1}Q^{-1} \\ &= N(Q^{-1} + N)^{-1}Q^{-1} = N(I + QN)^{-1}.\end{aligned}$$

This proves (4.5). \square

For a version of Lemma 4.3 with T just nonnegative, not necessarily strictly positive, see Lemma 2.10 in [11].

5. PROOF OF THEOREM 1.2

In this section, we will convert the infinite dimensional state space realization for the central solution X in (3.1) to the finite dimensional realization for X in (1.12), and in the mean time prove Theorem 1.2. Throughout G and K are the rational matrix functions described by the observable stable realization (1.3), and we assume that $T_G T_G^* - T_K T_K^*$ is strictly positive. Thus items (i) and (ii) in Theorem 1.1 are satisfied. In what follows we shall freely use the notations introduced in these two items. In particular, the operator T_R is strictly positive, Q is the stabilizing solution of the algebraic Riccati equation (1.10), and the matrix $\Delta = R_0 - \Gamma^* Q \Gamma$ is strictly positive. We set

$$(5.1) \quad C_0 = \Delta^{-1}(C - \Gamma^* Q A) \quad \text{and} \quad A_0 = A - \Gamma C_0.$$

By item (c) in Theorem 1.1 the matrix A_0 is stable. Using C_0 and A_0 in (5.1), the Riccati equation (1.10) can be rewritten as a Stein equation:

$$(5.2) \quad Q - A^* Q A_0 = C^* C_0.$$

The observability operator for the pair $\{C_0, A_0\}$ is the operator W_0 defined by

$$(5.3) \quad W_0 = \begin{bmatrix} C_0 \\ C_0 A_0 \\ C_0 A_0^2 \\ \vdots \end{bmatrix} : \mathbb{C}^n \rightarrow \ell_+^2(\mathbb{C}^m).$$

We need the following lemma (cf., identity (3.19) in [13]):

Lemma 5.1. *The operator W_0 is one-to-one, $T_R^{-1}W_{obs} = W_0$, and*

$$(5.4) \quad R(z)C_0(I_n - zA_0)^{-1} = C(I_n - zA)^{-1} + \Gamma^*(zI_n - A^*)^{-1}Q.$$

Proof. Let us assume that (5.4) has been proved. Note that the matrix function $C(I_n - zA)^{-1}$ is a stable rational function, while $\Gamma^*(zI_n - A^*)^{-1}Q$ is a rational function which is analytic on the exterior of the open unit disc and has the value zero at infinity. But then (5.4) implies that $T_R W_0 x$ is equal to $W_{obs} x$ for each $x \in \mathbb{C}^n$. Since T_R is invertible, we get $W_0 = T_R^{-1}W_{obs}$. Recall that W_{obs} is one to one. Therefore $W_0 = T_R^{-1}W_{obs}$ is also one to one.

It remains to prove (5.4). To do this we use the realization (1.6). Using $\Gamma C_0 = A - A_0$, a standard calculation shows that

$$zC(I_n - zA)^{-1}\Gamma C_0(I_n - zA_0)^{-1} = C(I_n - zA)^{-1} - C(I_n - zA_0)^{-1}.$$

Analogously, using the Stein equation (5.2), one computes that

$$\begin{aligned} \Gamma^*(zI_n - A^*)^{-1}C^*C_0(I_n - zA_0)^{-1} &= \Gamma^*(zI_n - A^*)^{-1}Q + \\ &\quad + \Gamma^*QA_0(I_n - zA_0)^{-1}. \end{aligned}$$

Using the realization (1.6) the two preceding identities yield

$$\begin{aligned} R(z)C_0(I_n - zA_0)^{-1} &= C(I_n - zA)^{-1} + \Gamma^*(zI_n - A^*)^{-1}Q + \\ &\quad + (-C + R_0C_0 + \Gamma^*QA_0)(I_n - zA_0)^{-1}. \end{aligned}$$

Next using the two identities in (5.1) we see that

$$\begin{aligned} -C + R_0C_0 + \Gamma^*QA_0 &= -C + R_0C_0 + \Gamma^*QA - \Gamma^*Q\Gamma C_0 \\ &= -(C - \Gamma^*QA) + (R_0 - \Gamma^*Q\Gamma)C_0 \\ &= -(C - \Gamma^*QA) + \Delta C_0 = 0. \end{aligned}$$

This proves (5.4). \square

Proof of Theorem 1.2. In the course of this proof we shall often use the following identity (which follows from (1.18) and Lemma 5.1):

$$(5.5) \quad W_{obs}^* W_0 = Q.$$

The Schur complement for T_R . From the realization (1.6) it follows that T_R admits a block 2×2 matrix representation

$$T_R = \begin{bmatrix} R_0 & \Gamma^*W_{obs}^* \\ W_{obs}\Gamma & T_R \end{bmatrix} \text{ on } \begin{bmatrix} \mathbb{C}^m \\ \ell_+^2(\mathbb{C}^m) \end{bmatrix}.$$

Since T_R is invertible, the *Schur complement* with respect to the $(2,2)$ entry is given by

$$R_0 - \Gamma^*W_{obs}^*T_R^{-1}W_{obs}\Gamma = R_0 - \Gamma^*Q\Gamma = \Delta.$$

It follows (see. e.g., [4, page 29]) that the inverse of T_R is given by

$$T_R^{-1} = \begin{bmatrix} \Delta^{-1} & -\Delta^{-1}\Gamma^*W_{obs}^*T_R^{-1} \\ -T_R^{-1}W_{obs}\Gamma\Delta^{-1} & T_R^{-1} + T_R^{-1}W_{obs}\Gamma\Delta^{-1}\Gamma^*W_{obs}^*T_R^{-1} \end{bmatrix} \text{ on } \begin{bmatrix} \mathbb{C}^m \\ \ell_+^2(\mathbb{C}^m) \end{bmatrix}.$$

A matrix representation for Ξ . To compute a finite dimensional realization for our central (maximum entropy) solution X in (3.1), we need a formula for Ξ involving the state space data. Note that (1.3) is equivalent to the following two realizations:

$$(5.6) \quad G(z) = D_1 + zC(I_n - zA)^{-1}B_1, \quad K(z) = D_2 + zC(I_n - zA)^{-1}B_2.$$

Using the realization for K in (5.6) and (1.11), we obtain

$$\begin{aligned} \Xi &= (T_G T_G^* - T_K T_K^*)^{-1} T_K E_q \\ &= T_R^{-1} T_K E_q + T_R^{-1} W_{obs} \Omega W_{obs}^* T_R^{-1} T_K E_q \quad [\text{by (1.11)}] \\ &= T_R^{-1} T_K E_q + W_0 \Omega W_0^* T_K E_q \quad [\text{by Lemma 5.1}] \\ &= T_R^{-1} T_K E_q + W_0 \Omega \begin{bmatrix} C_0^* & A_0^* W_0^* \end{bmatrix} \begin{bmatrix} D_2 \\ W_{obs} B_2 \end{bmatrix} \\ &= T_R^{-1} T_K E_q + W_0 \Omega (C_0^* D_2 + A_0^* Q B_2) \quad [\text{by (5.5)}] \\ &= T_R^{-1} T_K E_q + W_0 \Omega C_2^* \quad [\text{with } C_2 \text{ as in (1.13)}]. \end{aligned}$$

To compute $T_R^{-1} T_K E_q$, we use the 2×2 operator matrix representation of T_R^{-1} given above. This yields

$$\begin{aligned} T_R^{-1} T_K E_q &= \begin{bmatrix} \Delta^{-1} & -\Delta^{-1}\Gamma^*W_{obs}^*T_R^{-1} \\ -T_R^{-1}W_{obs}\Gamma\Delta^{-1} & T_R^{-1} + T_R^{-1}W_{obs}\Gamma\Delta^{-1}\Gamma^*W_{obs}^*T_R^{-1} \end{bmatrix} \begin{bmatrix} D_2 \\ W_{obs} B_2 \end{bmatrix} \\ &= \begin{bmatrix} \Delta^{-1} D_2 - \Delta^{-1} \Gamma^* Q B_2 \\ W_0 (B_2 + \Gamma \Delta^{-1} \Gamma^* Q B_2 - \Gamma \Delta^{-1} D_2) \end{bmatrix}. \end{aligned}$$

Substituting this into our previous formula for Ξ we arrive at

$$(5.7) \quad \Xi = \begin{bmatrix} \Delta^{-1} D_2 - \Delta^{-1} \Gamma^* Q B_2 + C_0 \Omega C_2^* \\ W_0 (B_2 + \Gamma \Delta^{-1} \Gamma^* Q B_2 - \Gamma \Delta^{-1} D_2 + A_0 \Omega C_2^*) \end{bmatrix} = \begin{bmatrix} D_0 \\ W_0 B_0 \end{bmatrix}.$$

See Theorem 1.2 for the definitions of D_0 and B_0 . Finally, note that the identity (5.7) also shows that $S_m^* \Xi = W_0 B_0$.

The state space operator F . The state space operator

$$(5.8) \quad F = S_m^* - S_m^* \Xi D_V^{-1} E_q^* T_K^* = S_m^* - W_0 B_0 D_V^{-1} E_q^* T_K^*.$$

The state space realizations for G and K in (5.6) yield

$$\begin{aligned} E_p^* T_G^* W_0 &= \begin{bmatrix} D_1^* & B_1^* W_{obs}^* \end{bmatrix} \begin{bmatrix} C_0 \\ W_0 A_0 \end{bmatrix} = D_1^* C_0 + B_1^* Q A_0 = C_1, \\ (5.9) \quad E_p^* T_K^* W_0 &= \begin{bmatrix} D_2^* & B_2^* W_{obs}^* \end{bmatrix} \begin{bmatrix} C_0 \\ W_0 A_0 \end{bmatrix} = D_2^* C_0 + B_2^* Q A_0 = C_2. \end{aligned}$$

Using the definition of C_2 with $S_m^* W_0 = W_0 A_0$, we obtain

$$F W_0 = S_m^* W_0 - W_0 B_0 D_V^{-1} E_q^* T_K^* W_0 = W_0 A_0 - W_0 B_0 D_V^{-1} C_2.$$

This leads to the following intertwining relation:

$$(5.10) \quad F W_0 = W_0 A^\times \quad \text{where} \quad A^\times = A_0 - B_0 D_V^{-1} C_2.$$

This readily implies that

$$(5.11) \quad (I - zF)^{-1} S_m^* \Xi = (I - zF)^{-1} W_0 B_0 = W_0 (I_n - zA^\times)^{-1} B_0.$$

Substituting the previous formulas into our state space formula for X in (3.1), we obtain

$$\begin{aligned}
 X(z) &= D_U D_V^{-1} + z (E_p^* T_G^* - D_U D_V^{-1} E_q^* T_K^*) (I - zF)^{-1} S_m^* \Xi D_V^{-1} \\
 &= D_U D_V^{-1} + z (E_p^* T_G^* - D_U D_V^{-1} E_q^* T_K^*) W_0 (I_n - zA^\times)^{-1} B_0 D_V^{-1} \\
 (5.12) \quad &= D_U D_V^{-1} + z (C_1 - D_U D_V^{-1} C_2) (I_n - zA^\times)^{-1} B_0 D_V^{-1}.
 \end{aligned}$$

Computing D_U and D_V . To complete our finite dimensional state space realization formula for X in (1.12), we need an expression for D_U and D_V , that is,

$$\begin{aligned}
 D_U &= E_p^* T_G^* \Xi = \begin{bmatrix} D_1^* & B_1^* W_{obs}^* \end{bmatrix} \begin{bmatrix} D_0 \\ W_0 B_0 \end{bmatrix} = D_1^* D_0 + B_1^* Q B_0 \\
 D_V &= I_q + E_q^* T_K^* \Xi = I_q + \begin{bmatrix} D_2^* & B_2^* W_{obs}^* \end{bmatrix} \begin{bmatrix} D_0 \\ W_0 B_0 \end{bmatrix} = I + D_2^* D_0 + B_2^* Q B_0.
 \end{aligned}$$

This together with (5.12) yields the finite dimensional state space formula for the maximum entropy solution X in (1.12).

It is noted that $V(0) = D_V$. So the entropy $\mathcal{E}(X) = -\ln \det[D_V]$; see (2.4).

The function V is invertible outer and A^\times is stable. Recall that $S_m^* \Xi = W_0 B_0$. Using this with $S_m^* W_0 = W_0 A_0$ and the state space realization for V in (3.8), we obtain

$$\begin{aligned}
 V(z) &= D_V + z E_q^* T_K^* (I - zS_m^*)^{-1} S_m^* \Xi \\
 &= D_V + z E_q^* T_K^* (I - zS_m^*)^{-1} W_0 B_0 \\
 &= D_V + z E_q^* T_K^* W_0 (I_n - zA_0)^{-1} B_0.
 \end{aligned}$$

Since $C_2 = E_q^* T_K^* W_0$, a finite dimensional realization for V is given by

$$(5.13) \quad V(z) = D_V + z C_2 (I_n - zA_0)^{-1} B_0.$$

Since A_0 is stable, the function V is a stable rational matrix function. In particular, V belongs to $H_{q \times q}^\infty$. From Theorem 2.1 we know that V^{-1} belongs to $H_{q \times q}^\infty$. Thus both V and V^{-1} are in $H_{q \times q}^\infty$, and so V is invertible outer.

Next we prove that A^\times is stable. By employing a standard state space inversion formula, the inverse for V given by

$$V(z)^{-1} = D_V^{-1} - z D_V^{-1} C_2 (I_n - zA^\times)^{-1} B_0 D_V^{-1}$$

where $A^\times = A_0 - B_0 D_V^{-1} C_2$. We already know that V^{-1} belongs to $H_{q \times q}^\infty$. Hence V^{-1} is also a stable rational matrix function. Because A_0 is stable and $V(z)^{-1}$ is analytic in the closed unit disc, Theorem 2.1 in [4] tells us that A^\times is stable.

The solution X is strictly contractive. It remains to show that $\|X\|_\infty < 1$. From Theorem 2.1 we know that $\Theta = V(0)V^{-1}$ is the outer spectral factor of $I - X^*X$. However, as proved in the preceding paragraph, the function V^{-1} is invertible outer. Hence Θ is invertible outer. The latter implies that $I - X^*(\zeta)X(\zeta)$ is strictly positive for each $\zeta \in \mathbb{T}$. Therefore $\|X(\zeta)\| < 1$ for $\zeta \in \mathbb{T}$. Thus X is a strictly contractive solution to our Leech problem. \square

Remark 5.2. For later purposes (see the next remark) we mention that U is given by the following finite dimensional realization:

$$(5.14) \quad U(z) = D_U + z C_1 (I_n - zA_0)^{-1} B_0.$$

The proof is similar to the proof of the realization of V in (5.13). Indeed, using $S_m^* W_0 = W_0 A_0$ with the state space realization for U in (3.7), we obtain

$$\begin{aligned} U(z) &= D_U + zE_p^* T_G^* (I - zS_m^*)^{-1} S_m^* \Xi \\ &= D_U + zE_p^* T_G^* (I - zS_m^*)^{-1} W_0 B_0 \\ &= D_U + zE_p^* T_G^* W_0 (I_n - zA_0)^{-1} B_0. \end{aligned}$$

Remark 5.3. Given the various matrices appearing in Theorems 1.1 and 1.2 one can now also prove directly that the function $X = UV^{-1}$ given by (1.12) satisfies $GX = K$, independent of the operator theory result based on the commutant lifting theorem. To illustrate this we give a direct proof of the identity $GU = KV$, using the realizations of U and V given by (5.14) and (5.13), respectively. The direct proof requires a number of non-trivial identities which are given by the following lemma.

Lemma 5.4. *Let $\begin{bmatrix} G & K \end{bmatrix}$ be given by the observable stable realization in (1.3), and assume that items (i) and (ii) in Theorem 1.1 are satisfied. Define $\Omega_0 = I + (P_2 - P_1)Q$. Then the following identities hold:*

$$(5.15) \quad B_1 C_1 - B_2 C_2 = A\Omega_0 - \Omega_0 A_0,$$

$$(5.16) \quad D_1 C_1 - D_2 C_2 = C\Omega_0,$$

$$(5.17) \quad B_1 D_U - B_2 D_V = -\Omega_0 B_0,$$

$$(5.18) \quad D_1 D_U - D_2 D_V = 0.$$

Here C_1 and C_2 are given by (1.13), and the matrices A_0 and C_0 are given by (5.1).

For the moment let us assume that the above identities are proved, and let us consider $G(z)U(z) - K(z)V(z)$. Using the realizations in (5.6), (5.14), and (5.13), we see that

$$\begin{aligned} G(z)U(z) - K(z)V(z) &= (D_1 D_U - D_2 D_V) + \\ &\quad + zC(I_n - zA)^{-1}(B_1 D_U - B_2 D_V) \\ &\quad + z(D_1 C_1 - D_2 C_2)(I_n - zA_0)^{-1} B_0 \\ &\quad + zC(I_n - zA)^{-1}(zB_1 C_1 - zB_2 C_2)(I_n - zA_0)^{-1} B_0. \end{aligned}$$

Now using the identity (5.15) we see that

$$\begin{aligned} (I_n - zA)^{-1}(zB_1 C_1 - zB_2 C_2)(I_n - zA_0)^{-1} &= \\ &= (I_n - zA)^{-1}(zA\Omega_0 - z\Omega_0 A_0)(I_n - zA_0)^{-1} \\ &= (I_n - zA)^{-1}(\Omega_0(I_n - zA_0) - (I_n - zA)\Omega_0)(I_n - zA_0)^{-1} \\ &= (I_n - zA)^{-1}\Omega_0 - \Omega_0(I_n - zA_0)^{-1}. \end{aligned}$$

It follows that

$$\begin{aligned} G(z)U(z) - K(z)V(z) &= (D_1 D_U - D_2 D_V) + \\ &\quad + zC(I_n - zA)^{-1}(B_1 D_U - B_2 D_V + \Omega_0 B_0) \\ &\quad + z(D_1 C_1 - D_2 C_2 - C\Omega_0)(I_n - zA_0)^{-1} B_0. \end{aligned}$$

The identities (5.18), (5.17), and (5.16) then show that $G(z)U(z) - K(z)V(z)$ is identically equal to zero, that is, $GU = KV$.

Proof of Lemma 5.4. In the sequel we shall use the following two identities

$$(5.19) \quad Q - A^*QA = C_0^*\Delta C_0 \quad \text{and} \quad Q - A^*QA_0 = C^*C_0.$$

These identities follow by using the definition of A_0 and C_0 together with the fact that Q is a hermitian matrix satisfying (1.10).

Proof of (5.15). Using $C_j = D_j^*C_0 + B_j^*QA_0$ for $j = 1, 2$ and the second Stein equation in (5.19), we have

$$\begin{aligned} B_jC_j &= B_jD_j^*C_0 + B_jB_j^*QA_0 = B_jD_j^*C_0 + (P_j - AP_jA^*)QA_0 \quad [\text{by (1.9)}] \\ &= B_jD_j^*C_0 + P_jQA_0 - AP_jA^*QA_0 \\ &= B_jD_j^*C_0 + P_jQA_0 - AP_j(Q - C^*C_0) \quad [\text{by the second part of (5.19)}] \\ &= (B_jD_j^* + AP_jC^*)C_0 + P_jQA_0 - AP_jQ \quad (j = 1, 2). \end{aligned}$$

Taking differences we obtain:

$$\begin{aligned} B_1C_1 - B_2C_2 &= (B_1D_1^* - B_2D_2^* + A(P_1 - P_2)C^*)C_0 + \\ &\quad + (P_1 - P_2)QA_0 - A(P_1 - P_2)Q \\ &= \Gamma C_0 + (P_1 - P_2)QA_0 - A(P_1 - P_2)Q \quad [\text{by (1.8)}] \\ &= A - A_0 + (P_1 - P_2)QA_0 - A(P_1 - P_2)Q \quad [\text{by (5.1)}] \\ &= A(I + (P_2 - P_1)Q) - (I + (P_2 - P_1)Q)A_0 = A\Omega_0 - \Omega_0A_0. \end{aligned}$$

Hence (5.15) holds.

Proof of (5.16). Again using $C_j = D_j^*C_0 + B_j^*QA_0$ for $j = 1, 2$ we have

$$\begin{aligned} D_1C_1 - D_2C_2 &= (D_1D_1^* - D_2D_2^*)C_0 + (D_1B_1^* - D_2B_2^*)QA_0 \\ &= (D_1D_1^* - D_2D_2^*)C_0 + (\Gamma^* - C(P_1 - P_2)A^*)QA_0 \quad [\text{by (1.8)}] \\ &= (D_1D_1^* - D_2D_2^*)C_0 + \Gamma^*QA_0 - C(P_1 - P_2)A^*QA_0 \\ &= (D_1D_1^* - D_2D_2^*)C_0 + \Gamma^*Q(A - \Gamma C_0) + \quad [\text{by (5.1)}] \\ &\quad - C(P_1 - P_2)(Q - C^*C_0) \quad [\text{by (5.19)}] \\ &= (D_1D_1^* - D_2D_2^* + C(P_1 - P_2)C^*)C_0 + \\ &\quad + \Gamma^*QA - \Gamma^*Q\Gamma C_0 - C(P_1 - P_2)Q \\ &= (R_0 - \Gamma^*Q\Gamma)C_0 + \Gamma^*QA - C(P_1 - P_2)Q \quad [\text{by (1.7)}] \\ &= C - \Gamma^*QA + \Gamma^*QA - C(P_1 - P_2)Q \quad [\text{by (5.1)}] \\ &= C(I + (P_2 - P_1)Q) = C\Omega_0. \end{aligned}$$

Thus (5.16) holds.

Proof of (5.17). To establish (5.17) we use that $B_0 = B_2 - \Gamma D_0 + A\Omega C_2^*$. This identity follows from

$$\begin{aligned} B_0 &= B_2 - \Gamma\Delta^{-1}(D_2 - \Gamma^*QB_2) + A_0\Omega C_2^* \\ &= B_2 - \Gamma\Delta^{-1}(D_2 - \Gamma^*QB_2) + (A - \Gamma C_0)\Omega C_2^* \\ (5.20) \quad &= B_2 - \Gamma D_0 + A\Omega C_2^*. \end{aligned}$$

Using (1.8) we see that

$$\begin{aligned}
 B_1 D_U - B_2 D_V + \Omega_0 B_0 &= \\
 &= -B_2 + (B_1 D_1^* - B_2 D_2^*) D_0 + (B_1 B_1^* - B_2 B_2^*) Q B_0 + \Omega_0 B_0 \\
 &= -B_2 + (\Gamma + A(P_2 - P_1) C^*) D_0 + (B_1 B_1^* - B_2 B_2^*) Q B_0 + \Omega_0 B_0 \\
 &= -B_2 + \Gamma D_0 + A(P_2 - P_1) C^* D_0 + (B_1 B_1^* - B_2 B_2^*) Q B_0 + \\
 &\quad + (I + (P_2 - P_1) Q) B_0 \\
 &= A \Omega C_2^* + A(P_2 - P_1) C^* D_0 + (B_1 B_1^* - B_2 B_2^*) Q B_0 + (P_2 - P_1) Q B_0,
 \end{aligned}$$

using (5.20) in the last identity. Next we use (1.9). This yields

$$\begin{aligned}
 B_1 D_U - B_2 D_V + (I + (P_2 - P_1) Q) B_0 &= \\
 &= A \Omega C_2^* + A(P_2 - P_1) C^* D_0 + \\
 &\quad + ((P_1 - P_2) - A(P_1 - P_2) A^*) Q B_0 + (P_2 - P_1) Q B_0 \\
 &= A \Omega C_2^* + A(P_2 - P_1) (C^* D_0 + A^* Q B_0).
 \end{aligned}$$

We proceed by computing $C^* D_0 + A^* Q B_0$. We have

$$\begin{aligned}
 C^* D_0 + A^* Q B_0 &= C^* D_0 + A^* Q B_2 - A^* Q \Gamma D_0 + A^* Q A \Omega C_2^* \quad [\text{by (5.20)}] \\
 &= (C^* - A^* Q \Gamma) D_0 + A^* Q B_2 + A^* Q A \Omega C_2^* \\
 &= C_0^* \Delta D_0 + A^* Q B_2 + A^* Q A \Omega C_2^*.
 \end{aligned}$$

Now observe that $C_0^* \Delta D_0 = C_0^* (D_2 - \Gamma^* Q B_2) + C_0^* \Delta C_0 \Omega C_2^*$. Using this together with the first identity in (5.19) we obtain

$$\begin{aligned}
 C^* D_0 + A^* Q B_0 &= C_0^* D_2 + (A^* - C_0^* \Gamma^*) Q B_2 + A^* Q A \Omega C_2^* + C_0^* \Delta C_0 \Omega C_2^* \\
 &= C_0^* D_2 + A_0^* Q B_2 + Q \Omega C_2^* \\
 &= C_2^* + Q \Omega C_2^* = (I_n + Q \Omega) C_2^*.
 \end{aligned}$$

Summarizing we have:

$$B_1 D_U - B_2 D_V + \Omega_0 B_0 = A \left(\Omega + (P_2 - P_1) + (P_2 - P_1) Q \Omega \right) C_2^*.$$

Now write Ω in (1.11) as $\Omega = -N(I + QN)^{-1}$, where $N = P_2 - P_1$. We see that

$$\begin{aligned}
 \Omega + (P_2 - P_1) + (P_2 - P_1) Q \Omega &= \\
 &= -N(I + QN)^{-1} + N + NQ \left(-N(I + QN)^{-1} \right) \\
 &= -N(I + QN)^{-1} + N - N(QN + I - I)(I + QN)^{-1} \\
 (5.21) \quad &= -N(I + QN)^{-1} + N(I + QN)^{-1} = 0.
 \end{aligned}$$

Therefore we obtain (5.17).

Proof of (5.18). To establish (5.18) notice that

$$\begin{aligned}
D_1 D_U - D_2 D_V &= D_1 (D_1^* D_0 + B_1^* Q B_0) - D_2 - D_2 (D_2^* D_0 + B_2^* Q B_0) \\
&= (D_1 D_1^* - D_2 D_2^*) D_0 + (D_1 B_1^* - D_2 B_2^*) Q B_0 - D_2 \\
&= (R_0 + C(P_2 - P_1)C^*) D_0 - D_2 + \quad [\text{use (1.7)}] \\
&\quad + (\Gamma^* + C(P_2 - P_1)A^*) (Q B_2 - Q \Gamma D_0 + Q A \Omega C_2^*) \quad [\text{use (1.8) and (5.20)}] \\
&= \left(R_0 - \Gamma^* Q \Gamma + C(P_2 - P_1)(C^* - A^* Q \Gamma) \right) D_0 + \\
&\quad + \left(\Gamma^* Q + C(P_2 - P_1)A^* Q \right) (B_2 + A \Omega C_2^*) - D_2 \\
&= \left(\Delta + C(P_2 - P_1)C_0^* \Delta \right) \left(\Delta^{-1} (D_2 - \Gamma^* Q B_2) + C_0 \Omega C_2^* \right) + \\
&\quad + \left(\Gamma^* Q + C(P_2 - P_1)A^* Q \right) (B_2 + A \Omega C_2^*) - D_2 \\
&= C(P_2 - P_1)C_0^* D_2 + C(P_2 - P_1) \left(A^* - C_0^* \Gamma^* \right) Q B_2 + \\
&\quad + \left(\Delta C_0 + C(P_2 - P_1)C_0^* \Delta C_0 + \Gamma^* Q A + C(P_2 - P_1)A^* Q A \right) \Omega C_2^* \\
&= C(P_2 - P_1)C_2^* + \\
&\quad + \left(\Delta C_0 + C(P_2 - P_1)Q + \Gamma^* Q A \right) \Omega C_2^* \quad [\text{by the first part of (5.19)}] \\
&= C \left(P_2 - P_1 + \Omega + (P_2 - P_1)Q \Omega \right) C_2^* = 0, \quad [\text{because of (5.21)}].
\end{aligned}$$

Therefore (5.18) holds. \square

As the identities in Lemma 5.4 show the matrices appearing in Theorems 1.1 and 1.2 have a lot of structure. As a further illustration of this fact we mention without proof the following identity:

$$C_1^* C_1 - C_2^* C_2 = \left(Q + Q(P_2 - P_1)Q \right) - A_0^* \left(Q + Q(P_2 - P_1)Q \right) A_0.$$

See also (5.4).

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DEPARTMENT OF AERONAUTICS AND ASTRONAUTICS, PURDUE UNIVERSITY, WEST LAFAYETTE, IN 47907, USA

E-mail address: `frazho@ecn.purdue.edu`

UNIT FOR BMI, NORTH-WEST UNIVERSITY, PRIVATE BAG X6001-209, POTCHEFSTROOM 2520, SOUTH AFRICA

E-mail address: `sanne.terhorst@nwu.ac.za`

DEPARTMENT OF MATHEMATICS, VU UNIVERSITY AMSTERDAM, DE BOELELAAN 1081A, 1081 HV AMSTERDAM, THE NETHERLANDS

E-mail address: `m.a.kaashoek@vu.nl`